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LINEARIZATION OF ACTIONS OF LOCALLY COMPACT GROUPS

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by

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ABSTRACT

In this paper we shall discuss the problem of linearization of actions of locally compact topological groups. Roughly, this concerns the following problem: given a group of homeomorphisms on a topological space, can this space be embedded in a topological vector space in such a way that the homeomorphisms from the given group become restrictions of linear homeomorphisms of the vector space? If the space under consideration is a Tychonov space, then the answer is "yes". There exist several constructions in the literature which prove this, and most of these turn out to be modifications of one single construction. This will be illustrated in Section 3 of this paper. In Section 2, a categorical framework for this basic construction will be evolved.

KEY WORDS & PHRASES: *topological transformation group, G-spaces linearization of G-spaces, categories of G-spaces.*

LINEARIZATION OF ACTIONS OF LOCALLY COMPACT GROUPS

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1. INTRODUCTION

In this paper we shall discuss the problem of linearization of actions of locally compact topological groups. Roughly, this concerns the following problem: given a group of homeomorphisms on a topological space, can this space be embedded in a topological vector space in such a way that the homeomorphisms from the given group become restrictions of linear homeomorphisms of the vector space?

1.1. Notational conventions. All notation is more or less standard. The identity mapping of a set X onto itself is denoted by 1_X. If f is a mapping, then inverse images are denoted by $\overleftarrow{f}[A]$, $\overleftarrow{f}(x)$, etc.
The expressions A := B and B =: A mean that A is defined to be B. If $\pi\colon X \times Y \to Z$ is a function then $\pi^x y := \pi(x,y) =: \pi_y(x)$ for every $(x,y) \in X \times Y$. So $\pi^x \in Z^Y$ for every $x \in X$ and $\pi_y \in Z^X$ for every $y \in Y$. The mappings $x \mapsto \pi^x\colon X \to Z^Y$ and $y \mapsto \pi_y\colon Y \to Z^X$ are denoted by $\bar{\pi}$ and $\underline{\pi}$, respectively; thus, $\bar{\pi}(x) := \pi^x$ and $\underline{\pi}(y) := \pi_y$ for $x \in X$ and $y \in Y$.
If a topological group G is given its discrete topology, this is indicated by writing G_d.
The unit element of a group G is always denoted by e.
For categorical notions, cf. MACLANE [1971]. In particular, the category of all topological spaces and continuous mappings will be denoted TOP.

1.2. Definitions. The following notions are basic in this paper. A *topological transformation group* (*ttg*) is a triple $\langle G,X,\pi\rangle$ where G is a topological group, X is a topological space and $\pi\colon G \times X \to X$ is a continuous mapping such that $\pi^e = 1_X$ and $\pi^s \circ \pi^t = \pi^{st}$ for all $s,t \in G$. So $\bar{\pi}: t \mapsto \pi^t$ is a homomorphism of G into the group of all autohomeomorphisms of X. The group G and the space X are called the *phase group* and the *phase space* of the ttg, and the mapping π is called the *action* of G on X. As a synonym for "phase space of a ttg with phase group G" the term G-*space* will be used; but sometimes we shall also call the ttg $\langle G,X,\pi\rangle$ a G-space. An *invariant subset* of a G-space $\langle G,X,\pi\rangle$ is a subset Y of X such that $\pi[G\times Y] \subseteq Y$; an *invariant point* is a point $x \in X$ such that $\{x\}$ is an invariant subset. If Y is an invariant subset of $\langle G,X,\pi\rangle$, then $\pi_Y := \pi|_{G\times Y}: G \times Y \to Y$ is an action of G on Y; the resulting ttg $\langle G,Y,\pi_Y\rangle$ is called the *restriction of* $\langle G,X,\pi\rangle$ *to* Y.

A *linear ttg*, or a *linear* G-*space*, is a ttg $\langle G,V,\pi\rangle$ in which V is a topological vector space and $\pi^t \in GL(V)$ for every $t \in G$, i.e., each π^t is a continuous linear operator on V with continuous inverse. *In this paper, all linear* G-*spaces will be assumed to be locally convex and separated.*

1.3. Examples.

1. If X is a topological space and G is a subgroup of the group of all autohomeomorphisms of X, then $\langle G_d,X,\delta\rangle$ is a ttg, where δ is the evaluation mapping, $\delta(\psi,x) := \psi(x)$ for $\psi \in G$ and $x \in X$. If X is a uniform space and G_u is the group G endowed with the topology of uniform convergence on X, then in the following cases $\langle G_u,X,\delta\rangle$ is a ttg: (a) G is equicontinuous, and (b) X is compact. See BOURBAKI [1971], Chapter X, or DE VRIES [1975c], Subsection 1.2, where also other topologies on homeomorphism groups are considered.

2. Suppose $f: X \to \mathbb{R}^n$ is a continuous vector field on an open subset X of $\mathbb{R}^n$ such that the initial value problem for the autonomous differential equation $\dot{x} = f(x)$ has the property that solutions exist, are unique and can be extended over $\mathbb{R}$. If $\pi_x: \mathbb{R} \to X$ denotes the solution with initial value $\pi_x(0) = x \in X$, then $\pi: (t,x) \mapsto \pi_x(t): \mathbb{R} \times X \to X$ is continuous, and $\langle R,X,\pi\rangle$ is a ttg (*global dynamical system*). For the topological study of problems arising from the qualitative theory of autonomous differential equations in the framework of ttg's (*topological dynamics*) the reader is referred to BHATIA & SZEGÖ [1970] or I.U. BRONŠTEIN [1979].

3. Let $C^\infty(\mathbb{R}^2,\mathbb{R}) := \{f: \mathbb{R}^2 \to \mathbb{R} \mid f$ is C^∞ and f and its partial derivatives of first order are bounded$\}$; for any function $f: \mathbb{R}^2 \to \mathbb{R}$,

define $\hat{f}(x_1,x_2) := f(x_1,x_2)\exp\frac{1}{2}(x_1+x_2)$, and let $C^\infty_\nu := \{\psi: \mathbb{R}^2 \to \mathbb{R} \mid \hat{\psi} \in C^\infty(\mathbb{R}^2,\mathbb{R})\}$. Then C^∞_ν with the uniform norm is a Banach space. Define $\tau: \mathbb{R} \times C^\infty_\nu \to C^\infty_\nu$ by $\tau(t,u)(x_1,x_2) := u(x_1+t,x_2+t)\exp[(x_1+x_2)t+t^2]$ for $t \in \mathbb{R}$, $u \in C^\infty_\nu$ and $(x_1,x_2) \in \mathbb{R}^2$. Then $<\mathbb{R},C^\infty_\nu,\tau>$ is a linear ttg; cf. CARLSON [1972b]. It is useful to observe that for every $u \in C^\infty_\nu$ the mapping $\tau^*_u: (t,x_1,x_2) \mapsto \tau(t,u)(x_1,x_2): \mathbb{R}^3 \to \mathbb{R}$ is the unique solution of the partial differential equation

$$\frac{\partial u}{\partial t} = (x_1+x_2)u + \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2},$$

satisfying the initial value condition $\tau^*_u(0,x_1,x_2) = u(x_1,x_2)$ for all $(x_1,x_2) \in \mathbb{R}^2$.

4. If G is a topological group and Y is a topological space, then let $C_c(G,Y)$ be the space of all continuous functions from G to Y, endowed with the compact-open topology. Define $\rho: G \times C_c(G,Y) \to C_c(G,Y)$ by $\rho^t f(u) := f(ut)$ for $f \in C_c(G,Y)$ and $t,u \in G$. Then $<G_d,C_c(G,Y),\rho>$ is a ttg, and if G is locally compact, then ρ is continuous on $G \times C_c(G,Y)$, so in that case $<G,C_c(G,Y),\rho>$ is a ttg. Observe, that if Y is a Tychonov space, then so is $C_c(G,Y)$. If V is a locally convex topological vector space then with pointwise defined linear operations, $C_c(G,V)$ is a locally convex topological vector space as well; it is complete with respect to its additive uniformity (which coincides with the uniformity of uniform convergence on compacta) provided V is complete and G is locally compact (cf. KELLEY [1955],p.231).

5. Let $C_c(G\times G)$ denote the space of all continuous functions from $G \times G$ to $\mathbb{R}$, endowed with the compact-open topology. If $r^t f(u,v) := f(ut,v)$ for $(t,f) \in G \times C_c(G\times G)$ and $(u,v) \in G \times G$, and if G is locally compact, then $<G,C_c(G\times G),r>$ is a linear G-space. Observe, that in a certain sense this linear G-space is related to the linear ttg of example 3 (for the case $G=\mathbb{R}$), because $<G,C_c(G\times G),r>$ is *isomorphic* with the linear G-space $<G,C_c(G\times G),\sigma>$, where $\sigma^t f(u,v) := f(ut,vt)$ for $(t,f) \in G \times C_c(G\times G)$ and $(u,v) \in G \times G$. Actually, an isomorphism is given by the mapping $f \mapsto f': C_c(G\times G) \to C_c(G\times G)$, where $f'(u,v) := f(u,vu^{-1})$ for $f \in C_c(G\times G)$ and $(u,v) \in G \times G$ (see Section 2 for the definition of (iso)morphism).

6. Let G be a locally compact Hausdorff group and fix a right Haar

measure μ on G (normalized by $\mu(G) = 1$ if G is compact). Let, for $1 \leq p < \infty$, $L^p(G)$ denote the space of (equivalence classes of μ-almost everywhere equal) μ-measurable functions $f: G \to \mathbb{R}$ such that $|f|^p$ is μ- integrable. Provided with the usual norm $\|\cdot\|_p: f \mapsto (\int_G |f|^p d\mu)^{1/p}: L^p(G) \to \mathbb{R}^+$, $L^p(G)$ is a Banach space. If we define $\rho: G \times L^p(G) \to L^p(G)$ similar as in Example 4 (right translation), then ρ is continuous, and $\langle G, L^p(G), \rho\rangle$ is a linear G-space; cf. DE VRIES [1975c], Subsection 2.3. In particular, $\langle G, L^2(G), \rho\rangle$ is a *unitary Hilbert G-space*, that is, a G-space whose phase space is a Hilbert space and where the action is by means of unitary operators. In this example, $\mathbb{R}$ can, of course, be replaced by $\mathbb{C}$ and even by a (real or complex) Hilbert space H. Thus, if $L^p(G,H)$ is the space of (equivalence classes of) weakly μ-measurable functions $f: G \to H$ such that $\int_G \|f\|^p d\mu$ exists and is finite, then $L^p(G,H)$ is a Banach space (in the case p=2 it is a Hilbert space) and $\langle G, L^p(G,H), \rho\rangle$ is a linear G-space.

7. Let G be a locally compact, sigma-compact Hausdorff topological group. Using results of PAALMAN-DE MIRANDA [1971] and DE VRIES [1972a;1979] it can be shown that there exists a function $w: G \to \mathbb{R}$ such that

(i) $w(t) > 0$ for every $t \in G$;

(ii) $w(st) \geq w(s)\, w(t)$ for every $(s,t) \in G \times G$;

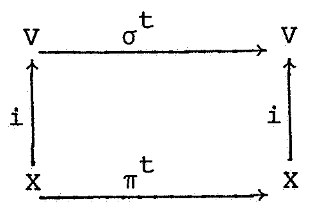
(iii) $w \in C_0(G) \cap \bigcap_{p\geq 1} L^p(G)$, i.e. w is continuous, $w(t) \to 0$ for $t \to \infty$ in G, and all powers of w are μ-integrable (here μ denotes right Haar measure on G).

A function with these properties will be called a (continuous) *weight function*. For a fixed weight function w on G, a regular Borel measure ν can be defined by $d\nu := w d\mu$, that is,

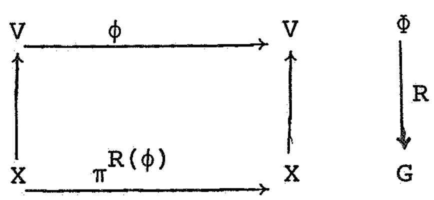
$$\int_G f(t)d\nu(t) := \int_G f(t)w(t)d\mu(t), \quad f \in C_{00}(G).$$

By condition (i) above, the support of ν is all of G. Let $L^2(G,\nu)$ denote the Hilbert space of all square ν-integrable functions on G (real valued). For $f \in L^2(G,\nu)$ and $t \in G$ it is easy to show that $\rho^t f \in L^2(G,\nu)$; here $\rho^t f$ is defined as in Examples 4 and 6. In fact, $\rho^t: L^2(G,\nu) \to L^2(G,\nu)$ is a bounded linear operator with $\|\rho^t\| \leq w(t)^{-1}$. Moreover, $\rho: G \times L^2(G,\nu) \to L^2(G,\nu)$ is continuous and $\langle G, L^2(G,\nu), \rho\rangle$ is a linear G-space (even a Hilbert G-space, but *not* a unitary Hilbert G-space); see DE VRIES [1975c],pp.74-77. In a similar way, we can define, for every Hilbert space H, the Hilbert G-space $\langle G,L^2(G,\nu;\mathcal{H}),\rho\rangle$. Here $L^2(G,\nu;\mathcal{H})$ is the Hilbert space of all weakly Borel measurable functions $f\colon G \to \mathcal{H}$ such that $\|f\|_2 := (\int_G \|f(t)\|^2 d\nu(t))^{\frac{1}{2}}$ exists and is finite.

1.4. We can now formulate our problem more precisely. We shall say that a ttg $\langle G,X,\pi\rangle$ *has a linearization* whenever it can be obtained as the restriction of a linear G-space $\langle G,V,\sigma\rangle$ to an invariant subset; that is, whenever there exists a topological embedding $i\colon X \to V$ such that for every $t \in G$ the the following diagram commutes:



The question is: which ttg's have actually linearizations? In this form, the linearization problem has been considered by many authors, e.g.BEBUTOV (cf. NEMYCKIĬ [1949], MOSTOV [1957], PALAIS [1961], DE GROOT [1962], BAAYEN & DE GROOT [1968], EDELSTEIN [1970], JANOS [1970], MANES [1972], to mention only a few (see also Section 3). The problem can also be considered in a slightly different form: under which conditions can the phase space X of a ttg $\langle G,X,\pi\rangle$ be embedded in a linear space V in such a way that, if i denotes the embedding mapping, $\{\pi^t \mid t \in G\} = \{\overleftarrow{i}\circ\phi\circ i \mid \phi \in \Phi\}$ for some subgroup Φ of $\mathcal{GL}(V)$; if so, can we obtain a commutative diagram



where $R\colon \Phi \to G$ is a homomorphism? This point of view has been worked out in BAAYEN [1964] and DE VRIES [1975c], Subsections 6.1 and 6.8, and we shall not consider it here.

The following theorem gives a necessary and sufficient condition for a G-space to have a linearization. The result appears explicitly in HAJEK [1971], DE VRIES [1975c], and SMIRNOV [1976]; it is implicit in almost all papers cited in the references. In Section 2 we shall give a categorical framework for the proof of the following theorem, and in Section 3 we shall discuss some modifications of the proof, leading to various results.

1.5. THEOREM. *Let* G *be a locally compact Hausdorff group. Then the following conditions are equivalent for a* G*-space* $\langle G,X,\pi\rangle$:

(i) *There exist a linear* G*-space* $\langle G,V,\sigma\rangle$ *and a topological embedding* $i: X \to V$ *such that* $\sigma^t \circ i = i \circ \pi^t$ *for all* $t \in G$.

(ii) X *can be embedded in a topological vector space.*

(iii) X *is a Tychonov space.*

PROOF. (i) $\Rightarrow$ (ii) $\Longleftrightarrow$ (iii): obvious or well-known.

(ii) $\Rightarrow$ (i): The mapping

$$\underline{\pi}: x \mapsto \pi_x : X \to C_c(G,X)$$

is a topological embedding (cf. e.g. DE VRIES [1975c], 2.1.13). Let $f: X \to V_0$ be an embedding of X in a topological vector space V_0. Then $h \mapsto f\circ h: C_c(G,X) \to C_c(G,V_0)$ is a topological embedding (cf. Example 4 above for notation), and so is

$$i: x \mapsto f\circ\pi_x : X \to C_c(G,V_0).$$

Now $\langle G,C_c(G,V_0),\rho\rangle$ is a linear G-space (cf. Example 4) and it is easily checked that $\rho^t\circ i = i\circ\pi^t$ for all $t \in G$. □

2. CATEGORIES OF G-SPACES

In this section, G *shall always be a locally compact Hausdorff group.*

2.1. If $\langle G,X,\pi\rangle$ and $\langle G,Y,\sigma\rangle$ are G-spaces, then an *equivariant mapping* from X to Y is a mapping $f: X \to Y$ such that $f\circ\pi^t = \sigma^t\circ f$ for all $t \in G$ [)1].

)1 If this relation holds only for all t in a subgroup H of G, then f is called an H-*equivariant mapping*.

Continuous equivariant mappings will be called *morphisms of G-spaces*. These morphisms, together with the class of all G-spaces, form in an obvious way a category, which will be denoted TOP^G. Two G-spaces are *isomorphic* if they are isomorphic in this category (i.e., if and only if there exists an equivariant homeomorphism between them).

There is an obvious functor S^G: $\mathrm{TOP}^G \to \mathrm{TOP}$, the *phase space functor*, defined by

$$S^G: \begin{cases} \langle G,X,\pi\rangle \mapsto X \text{ on objects} \\ f \mapsto f \text{ on morphisms (forgetting the property of being equivariant).} \end{cases}$$

If K is a subcategory of TOP, then $K^G := (S^G)^{\leftarrow}[K]$. Then K^G is a subcategory of TOP^G, and K^G is a full subcategory of TOP^G if and only if K is full in TOP. We shall consider only the following subcategories of TOP (which were not considered in DE VRIES [1975c]):

CR: the full subcategory of TOP, determined by the class of all Tychonov spaces;

LCV: objects: all locally convex Hausdorff topological vector spaces; morphisms: all continuous linear maps between such spaces;

CLCV: the full subcategory of LCV, determined by the class of all *complete* locally convex Hausdorff topological vector spaces.

However, we are not so much interested in the categories LCV^G and CLCV^G: we want to consider *linear* G-spaces. So let $\mathrm{l\text{-}LCV}^G$ and $\mathrm{l\text{-}CLCV}^G$ denote the full subcategories of LCV^G and CLCV^G, respectively, determined by the classes of all *linear G-spaces* and all *complete linear G-spaces*, respectively.

2.2. We shall give now a brief review of the relevant parts of DE VRIES [1975c], Subsection 3.2 and 6.3. The results, formulated there for TOP^G, apply also to its subcategory CR^G and we shall formulate them only for the latter.

The functor S^G: $\mathrm{CR}^G \to \mathrm{CR}$ has a left adjoint. In fact, it is even a monadic functor. It follows, that S^G creates and preserves limits and monomorphisms. Since CR is complete, this implies that CR^G is a complete category.

For example, products in CR^G are just cartesian products of the phase spaces with coordinate-wise action of G. Monomorphisms in CR^G are continuous equivariant injections. (For these results, local compactness of G is irrelevant.)

As G is locally compact, the functor S^G has also a right adjoint M^G: $CR \to CR^G$. It is given by

$$M^G: \begin{cases} X \mapsto \langle G, C_c(G,X), \rho\rangle & \text{on objects} \\ f \mapsto f\circ - & \text{on morphisms,} \end{cases}$$

where for $f\colon X \to Y$ (in CR) the morphism $f\circ-$ in CR^G is defined by $f\circ-\colon h \mapsto f\circ h\colon C_c(G,X) \to C_c(G,Y)$. The unit of the adjunction of S^G and M^G is given by the universal arrows

$$\langle G,X,\pi\rangle \xrightarrow{\;\underline{\pi}\;} \langle G,C_c(G,X),\rho\rangle = M^G S^G \langle G,X,\pi\rangle$$

for objects $\langle G,X,\pi\rangle$ in CR^G, and the counit is given by the arrows

$$S^G M^G Y = C_c(G,Y) \xrightarrow{\;\delta_e\;} Y$$

for objects Y in CR (δ_e denotes evaluation-at-e). Recall, that the characterizing property of the counit is the following: for every object $\langle G,X,\pi\rangle$ in CR^G and every continuous function $f\colon X \to Y$ there exists a unique morphism $\tilde{f}\colon \langle G,X,\pi\rangle \to \langle G,C_c(G,Y),\rho\rangle$ such that $\delta_e\circ\tilde{f} = f$.

$$\begin{array}{ccc} \langle G,C_c(G,Y),\rho\rangle & \qquad & C_c(G,Y) \xrightarrow{\;\delta_e\;} Y \\ \uparrow \tilde{f} & & \tilde{f}\uparrow \quad \nearrow f \\ \langle G,X,\pi\rangle & & X \end{array}$$

In fact, $\tilde{f}$ is given by $\tilde{f} := f\circ\underline{\pi}$. It follows that if $f\colon X \to Y$ is a topological embedding, then $\tilde{f}\colon X \to C_c(G,Y)$ is a topological (and equivariant) embedding: cf. the proof of Theorem 1.5.

2.3. REMARKS.

1. In SMIRNOV [1976], Thm.5, it has been shown that in the above situation, $\tilde{f}$ is a *closed* embedding whenever f is a closed embedding and G is compact.

2. If we denote the set of all morphisms of G-spaces from <G,X,π> to <G,C_c(G,Y),ρ> by $Mor^G(X,C_c(G,Y))$, then the adjunction of S^G and M^G described above implies that

$$f \mapsto \tilde{f} = f\circ\pi\colon C(X,Y) \to Mor^G(X,C_c(G,Y))$$

is a bijection which has as its inverse the mapping $\tilde{f} \mapsto \delta_e\circ\tilde{f}$. By elementary results about function spaces (cf. DUGUNDJI [1966], XII.2.1), *both mappings are continuous if* C(X,Y) *and* $Mor^G(X,C_c(G,Y))$ *are given the compact-open topology*. This fact has been observed, among others, in MAXWELL [1966] and SMIRNOV [1976],Thm.7.

3. The adjunction of S^G and M^G is also related to results about extensors for G-spaces (or cogenerators in a more categorical language); cf. DE VRIES [1975c], Subsection 6.4 and also DE VRIES [1979b].

4. The functor S^G has not only a right adjoint, it is comonadic. So S^G preserves and creates colimits and epimorphisms. In particular, epimorphisms in CR^G are just the equivariant continuous mappings having a dense range.

2.4. LEMMA. *The functor* M^G *sends* LCV *into* 1-LCV^G *and* CLCV *into* 1-$CLCV^G$. *Moreover, the restricted functors*

$$M^G\colon LCV \to 1\text{-}LCV^G \quad \textit{and} \quad M^G\colon CLCV \to 1\text{-}CLCV^G$$

are right adjoint to the restricted functors

$$S^G\colon 1\text{-}LCV^G \to LCV \quad \textit{and} \quad S^G\colon 1\text{-}CLCV^G \to CLCV,$$

respectively.

PROOF. The first statement is easily checked (cf. also the observations in Example 4 of Section 1). The second statement is an immediate consequence of the first and the observation that the universal arrows $\underline{\pi}$ and δ_e in 2.3 are *linear* if $\langle G,X,\pi\rangle$ is a linear ttg and Y is a linear space. □

2.5. We shall discuss now a linearization process which is completely different from the construction suggested in the proof of Theorem 1.5, although for the proof of its correctness we need the functor M^G.

The following is well-known. The category **LCV** is a reflective subcategory of **CR**, the reflector being the "free topological vector space functor" F: **CR** $\to$ **LCV**. For every object X in **CR**, FX is the free locally convex Hausdorff topological vector space, generated by X. The canonical mapping $i_X: X \to FX$ (reflection of X in **LCV**) has the following properties:

(i) For every continuous function $f: X \to V$, where V is an arbitrary object in **LCV**, there exists a unique continuous linear mapping $f: FX \to V$ such that $f = \bar{f}\circ i_X$.

(ii) $i_X: X \to FX$ is a closed embedding, and the linear hull of $i_X[X]$ is just all of FX.

Next, observe that **CLCV** is a reflective subcategory of **LCV**, the reflector of **LCV** into **CLCV** being the "completion functor" (in fact, **CLCV** is the reflective subcategory of **LCV**, generated by the full subcategory of all Banach spaces). Thus, if $\tilde{F}X$ denotes the completion of X, then the composite functor $\tilde{F}$: **CR** $\to$ **CLCV** is a reflector, and the mappings $i_X: X \to \tilde{F}X$ (we consider, for convenience, FX just as a subspace of $\tilde{F}X$) have the following properties:

(iii) The same universal property as described in (i) above, but now for objects V in **CLCV**.

(iv) $i_X: X \to \tilde{F}X$ is a topological embedding and the linear hull of $i_X[X]$ is dense in $\tilde{F}X$.

We shall show now, that similar results are valid for the corresponding categories of G-spaces.

2.6. LEMMA. *The inclusion functors*

$$1\text{-}\mathbf{CLCV}^G \to 1\text{-}\mathbf{LCV}^G \to \mathbf{CR}^G$$

all have left adjoints, that is, 1-CLCV^G *is a reflective subcategory of* 1-LCV^G *and* 1-LCV^G *is a reflective subcategory of* CR^G.

PROOF. Straightforward consequence of general criteria (cf. e.g. HERRLICH & STRECKER [1973],37.1). For the application of these criteria, the description of products, monomorphisms and epimorphisms in CR^G as given in 2.2 and 2.3(4) above are useful. □

Our next lemma will be the tool for investigating the reflections of objects of CR^G into 1-LCV^G and 1-CLCV^G. It is related to the proof that the adjoint of a composition of functors is the composition of adjoints; for the rather straightforward proof, cf. DE VRIES [1975c],4.4.10.

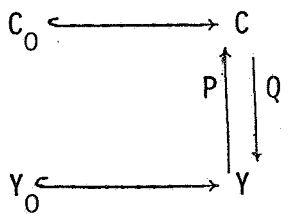
2.7 LEMMA. *Let* Y *and* C *be categories,* Y_0 *and* C_0 *reflective subcategories of* Y *and* C*, respectively. Suppose we are given a functor* $Q\colon C \to Y$ *having a left adjoint* $P\colon Y \to C$ *with unit* $\alpha\colon 1_Y \to QP$ *and counit* $\beta\colon PQ \to 1_C$ *such that*

(i) $P[Y_0] \subseteq C_0$ *and for each object* c *in* C_0*, the universal arrow* $\beta_C\colon PQc \to c$ *is in* C_0.

(ii) $Q[C_0] \subseteq Y_0$ *and for each object* Y *in* Y_0 *the arrow* $\alpha_Y\colon Y \to QPY$ *is in* Y_0.

Then the functor P *preserves reflections, i.e., if* Y *is an object in* Y *and* $\rho_Y\colon Y \to FY$ *is its reflection in* Y_0*, then* $P\rho_Y\colon PY \to PFY$ *is the reflection of* PY *in* C_0. □

The following diagram illustrates the situation:



2.8. THEOREM. *Let* $\langle G,X,\pi\rangle$ *be an object in* CR^G*. Then its reflection in* 1-LCV^G *has the form*

$$i_X\colon \langle G,X,\pi\rangle \to \langle G,FX,\pi^*\rangle$$

where $i_X: X \to FX$ *is the canonical closed embedding of* X *into the free locally convex Hausdorff topological vector space over* X*.*

PROOF. Apply 2.7 to the following situation

$$\begin{array}{ccc} \mathsf{LCV} & \hookrightarrow & \mathsf{CR} \\ & & S^G \uparrow\downarrow M^G \\ 1\text{-}\mathsf{LCV}^G & \hookrightarrow & \mathsf{CR}^G \end{array}$$

and observe that by 2.4 and its proof the conditions (i) and (ii) of 2.7 are fulfilled. □

2.9. COROLLARY. *Let* $\langle G,X,\pi\rangle$ *be a ttg with* X *a Tychonov space, and let* $i_X: X \to FX$ *be the canonical embedding of* X *in the free locally convex Hausdorff topological vector space over* X*. If for every* $t \in G$ *the canonical continuous linear extension of* $\pi^t: X \to X$ *over* FX *is denoted by* $\pi^{*t}: FX \to FX$*, then* $\pi^*: (t,\xi) \mapsto \pi^{*t}\xi: G \times FX \to FX$ *is continuous, and* $\langle G,FX,\pi^*\rangle$ *is a linear G-space in which* $\langle G,X,\pi\rangle$ *is equivariantly embedded as a closed invariant subset.*

PROOF. This is just a reformulation of 2.8. □

REMARK. A direct proof of 2.9, not using categorical methods, can be given along the lines of EISENBERG [1969]. However, also in that proof the linear G-space $\langle G,C_c(G,V_0),\rho\rangle$, used in the proof of Theorem 1.5 above, plays a role. Similar methods to obtain linearizations of mappings have been used in MANES [1972].

2.10. THEOREM. *Let* $\langle G,V,\pi\rangle$ *be an object in* $1\text{-}\mathsf{LCV}^G$*. Then its reflection in* $1\text{-}\mathsf{CLCV}^G$ *has the form*

$$j_V: \langle G,V,\pi\rangle \longrightarrow \langle G,\tilde{V},\tilde{\pi}\rangle$$

where $j_V: V \to \tilde{V}$ *is the canonical embedding of* V *into its completion.*

PROOF. Apply 2.7 to the following situation

$$\begin{array}{ccc} \mathrm{CLCV} & \hookrightarrow & \mathrm{LCV} \\ & & S^G \uparrow \downarrow M^G \\ 1\text{-}\mathrm{CLCV}^G & \hookrightarrow & 1\text{-}\mathrm{LCV}^G \end{array} \quad . \quad \square$$

2.11. COROLLARY. *Let* $<G,V,\pi>$ *be a linear* G-*space and* $j_V: V \to \tilde{V}$ *the canonical embedding of* V *in its completion. For every* $t \in G$, *let* $\tilde{\pi}^t: \tilde{V} \to \tilde{V}$ *denote the continuous linear extension of* $\pi^t: V \to V$. *Then* $\tilde{\pi}: (t,\xi) \mapsto \tilde{\pi}^t\xi: G \times \tilde{V} \to \tilde{V}$ *is continuous and* $<G,\tilde{V},\tilde{\pi}>$ *is a complete linear* G-*space in which* $<G,V,\pi>$ *is equivariantly embedded as a dense invariant linear subspace.*

PROOF. Clear from 2.10. $\square$

2.12. REMARK. Our proof of 2.11 is, in fact, a categorical "mystification" of the following simple argument, which is very close to the proof of Theorem 1.5. Consider the equivariant embedding $\xi \mapsto j_V \circ \pi_\xi: <G,V,\pi> \to$ $\to <G,C_c(G,\tilde{V}),\rho>$. Here $C_c(G,\tilde{V})$ is complete, hence the closure $\bar{V}$ of the image of V in $C_c(G,\tilde{V})$ is complete (and locally convex, of course). Since $\bar{V}$ is an invariant subspace, the restricted G-space $<G,\bar{V},\rho_{\bar{V}}>$ is a complete linear G-space in which $<G,V,\pi>$ is densely and equivariantly embedded. Since *every* dense linear embedding of V into a complete locally convex vector space is isomorphic to the completion $\tilde{V}$ of V, this proves 2.11.

2.13. We leave it to the reader to combine 2.8 and 2.10 in order to find a description of the reflection of an object of CR^G into 1-CLCV^G.

3. SOME PARTICULAR LINEARIZATIONS

In 1.5 and 2.9 we have presented two different constructions which prove that Tychonov G-spaces have linearizations, provided G is locally compact. In this section we shall present some modifications of our first construction. In this way we are able to review most of the linearizing constructions which have been given in the literature.

In this section, $\langle G,X,\pi\rangle$ *shall always denote a G-space with* G *locally compact Hausdorff and* X *a Tychonov space.* Our aim is not only to construct a linear G-space for every G-space $\langle G,X,\pi\rangle$ separately, but to obtain "universal" G-spaces in which all members of a class of G-spaces can equivariantly be embedded.

3.1. Our starting point is the last remark in 2.2: every embedding $f\colon X \to V$ in a locally convex Hausdorff vector space induces an equivariant embedding of $\langle G,X,\pi\rangle$ in a linear G-space, viz.

$$x \mapsto f\circ\pi_x : \langle G,X,\pi\rangle \to \langle G,C_c(G,V),\rho\rangle .$$

We shall mention now some straightforward applications of this observation.

3.2. *If* X *is a separable metric space, then* X *can be embedded in the Hilbert space* ℓ^2, *so* $\langle G,X,\pi\rangle$ *can equivariantly be embedded in* $\langle G,C_c(G,\ell^2),\rho\rangle$. For the special case that $G = \mathbb{R}$ and X is a separable metrizable locally compact space, this result appears in TOADER [1974] as a consequence of a generalization of the BEBUTOV-KAKUTANI-HAJEK theorem; cf. 3.11 below. In WEST [1968], the above mentioned result has been applied to a compact matrizable G-space X in order to prove that every action of G on a subset of the Hilbert cube Q with property Z can be extended to an action of G on Q. Since every compact metric space can be embedded in Q as a set with property Z, it follows that *for every compact metric G-space* $\langle G,X,\pi\rangle$ *there exists an action* π' *of* G *on* Q *such that* $\langle G,X,\pi\rangle$ *can equivariantly be embedded in* $\langle G,Q,\pi'\rangle$ *as an invariant subset (having property* Z*).* Actually, local compactness of G is not necessary for this result. For a "universal" version of this result, see 3.6 below.

3.3. *If* κ *is a cardinal number and* X *has weight* $w(X) \le \kappa$, *then* X *can be embedded in* $\mathbb{R}^\kappa$, *hence* $\langle G,X,\pi\rangle$ *can equivariantly be embedded in* $\langle G,C_c(G,\mathbb{R}^\kappa),\rho\rangle$. Observe, that this embedding can be realized by the mapping $x \mapsto (g\circ\pi_x)_{g\in\mathcal{F}}\colon X \to C_c(G,\mathbb{R}^\kappa) \approx C_c(G,\mathbb{R})^\kappa$, where $\mathcal{F}$ is a collection of continuous functions from X into $\mathbb{R}$ which separates points and closed subsets of X, the cardinality of $\mathcal{F}$ being κ.

In DE VRIES [1978] it was shown that F can be choosen according to these conditions so that for every $g \in F$ the collection $\{g \circ \pi_x \mid x \in X\}$ is an equicontinuous set of functions from G to [0,1]. Then the image of X in $C_c(G,\mathbb{R}^\kappa)$ is actually an equicontinuous subset of $C_c(G,[0,1]^\kappa)$. Hence by ASCOLI's theorem, it has a compact closure Y in $C_c(G,[0,1]^\kappa)$. Then Y is an invariant subset of $C_c(G,\mathbb{R}^\kappa)$, and taking for σ the restriction of ρ to $G \times Y$, we have proved:

3.4. <u>THEOREM</u>. *Every* G-*space* $\langle G,X,\pi\rangle$ *with* G *a Tychonov space and* G *locally compact can equivariantly be embedded in a* G-*space* $\langle G,Y,\sigma\rangle$ *with* Y *a compact Hausdorff space of weight* $w(Y) \leq \max\{w(G),w(X)\}$. *In particular, if* G *and* X *are separable and metrizable, then so is* Y.

<u>PROOF</u>. Observe, that in the proof outlined above we have

$$w(Y) \leq w(C_c(G,\mathbb{R}^\kappa)) = \max\{w(C_c(G,\mathbb{R})),\kappa\}$$

for every $\kappa \geq w(X)$. Moreover, it is well-known that $w(C_c(G,\mathbb{R})) = w(G)$. □

<u>REMARK</u>. The above theorem appears in DE VRIES [1978]. It generalizes results of DE GROOT & MACDOWELL [1960]. For a completely different proof, cf. DE VRIES [1977].

3.5. <u>COROLLARY 1</u>. *For every cardinal number* $\kappa \geq w(G)$ *there exists a compact Hausdorff* G-*space* $\langle G,Y,\sigma\rangle$ *such that* $w(Y) = \kappa$ *in which every Tychonov* G-*space* $\langle G,X,\pi\rangle$ *with* $w(X) \leq \kappa$ *can equivariantly be embedded.*

<u>PROOF</u>. Apply 3.4 to the G-space $\langle G,C_c(G,\mathbb{R}^\kappa),\rho\rangle$. □

3.6. <u>COROLLARY 2</u>. *Let* G *be a separable metrizable locally compact group. There exists an action* σ *of* G *on the Hilbert cube* Q *such that every* G-*space* $\langle G,X,\pi\rangle$ *with* X *separable and metrizable can equivariantly be embedded in* $\langle G,Q,\sigma\rangle$.

<u>PROOF</u>. Apply WEST's result, quoted in 3.1, to the compact G-space $\langle G,Y,\sigma\rangle$ from 3.5. □

3.7. REMARK. Corollary 2 generalizes earlier results which were applicable only to countable groups; see e.g. BAAYEN [1964], Section 3.4. For other interesting actions of groups on the Hilbert cube, cf. ANDERSEN [1977].

3.8. COROLLARY 3. *Every Tychonov G-space* $\langle G,X,\pi\rangle$ *can equivariantly be embedded in a G-space* $\langle G,Y,\pi\rangle$ *where* Y *is a compact convex subset of a locally convex Hausdorff topological vector space, and* σ *is an affine action (i.e. each* σ^t *is an affine mapping of* Y *into itself).*

PROOF. In view of Theorem 3.4 we may assume that X is a compact invariant subset of a linear G-space $\langle G,V,\sigma\rangle$, where V is a *complete* locally convex Hausdorff topological vector space (cf. also the remarks in Example 4 of Section 1). It follows that the closed convex hull Y of X in V is compact.[)1] Since Y is easily seen to be invariant, the restriction of $\langle G,V,\sigma\rangle$ to Y is the desired compact convex affine G-space. □

3.9. REMARK. The idea of proof of Corollary 3 was used in WEST [1968]. Alternative proof: let $M(X)$ be the space of all regular Borel measures on X (X may assumed to be compact in view of 3.4), and let Y be the closed convex hull of $\delta[X]$ in $M(X)$, where $\delta\colon X \to M(X)$ is the canonical embedding of X in $M(X)$. As a corollary of ALAOGLU's theorem, Y is compact. The action of G on X induces a linear action of G_d on $M(X)$, and it is quite easy to see that the restriction of this action to the (compact!) set Y is an action of G on Y. Obviously, this action is affine.

If G is compact and X is separable metric then in 3.8, Y may assumed to be a compact convex subset of a Banach space (e.g. $C_c(G,\ell^2) = C_u(G,\ell^2)$; cf.3.2). This result is related to JAWOROWSKI [1976], Prop.4.1.

The remainder of this section is devoted to the following question: can the equivariant embedding of $\langle G,X,\pi\rangle$ in $\langle G,C_c(G,\mathbb{R}^\kappa),\rho\rangle$ be modified so as to obtain an embedding in a "smaller" or "nicer" linear G-space? The problem is best illustrated by its solutions, some of which we shall present now. The first one is the well-known BEBUTOV-KAKUTANI theorem. The original result of BEBUTOV (cf. NEMYCKIĬ [1949]) applies to the case that X is a compact metric $\mathbb{R}$-space, in which the set S of all invariant points consists

)1 Cf. for Example BOURBAKI [1953], Chap.I, §4, no. 1.

of at most one point. In KAKUTANI [1968], a new proof of BEBUTOV's result was given, which applied to compact metric $\mathbb{R}$-spaces in which S is homeomorphic with a subset of $\mathbb{R}$. In HAJEK [1971] the result has been generalized to the form which we shall present below, although his result was still restricted to the case that $G = \mathbb{R}$. The observation that $\mathbb{R}$ can be replaced by an arbitrary connected Lie group is due to CHEN [1975] (he made his observation only for KAKUTANI's proof, but it applies equally well to HAJEK's).

3.10. <u>THEOREM</u>. *Let* $\langle G,X,\pi\rangle$ *be a* G*-space with* G *a non-trivial connected Lie group and* X *a locally compact separable metric space. Then* $\langle G,X,\pi\rangle$ *can equivariantly be embedded in the linear* G*-space* $\langle G,C_c(G,\mathbb{R}),\rho\rangle$ *as a closed nowhere dense invariant subset if and only if the set* S *of invariant points in* X *is homeomorphic with a closed subset of* $\mathbb{R}$.

<u>PROOF</u>. "Only if": obvious, as the set of invariant points in $\langle G,C_c(G,\mathbb{R}),\rho\rangle$ is homeomorphic with $\mathbb{R}$ (it consists of all constant functions).

"If" (outline): it is sufficient to construct a continuous function $f: X \to \mathbb{R}$ such that

(i) for all $x,y \in X$, $x \neq y$, there exists $t \in G$ such that $f(\pi^t x) \neq f(\pi^t y)$, i.e. $f\circ\pi_x \neq f\circ\pi_y$.

(ii) $f(x) \to \infty$ in $\mathbb{R}$ as $x \to \infty$ in X (that is, for all $n \in \mathbb{R}$ the set $f^{\leftarrow}[-n,n]$ is compact in X).

Indeed, condition (i) implies that the function $F: x \mapsto f\circ\pi_x : X \to C_c(G,\mathbb{R})$ is a continuous *injection*, and then condition (ii) implies that F is a closed embedding. Since X is locally compact, $F[X]$ can have no interior points in the topological vector space $C_c(G,\mathbb{R})$, otherwise $C_c(G,\mathbb{R})$ would be locally compact, hence finite dimensional by a well-known result from functional analysis; but this would contradict connectedness and non-triviality of G.

In order to prove that $f: X \to \mathbb{R}$ with properties (i) and (ii) exists, first construct a continuous extension $f_0: X \to \mathbb{R}$ of the given homeomorphism h from the set S of invariant points of X onto a closed subset of $\mathbb{R}$; using TIETZE's extension lemma this can be done in such a way that $f_0(x) \to \infty$ as $x \to \infty$; cf. HAJEK [1971], Lemma 6. Thus, the set $\mathcal{F}$ of all continuous extensions of h having property (ii) is a non-empty closed subset of $C_u(X,\mathbb{R})$

(i.e. $C(X,\mathbb{R})$ endowed with the *uniform* topology). Then F is a complete metric space, and an application of BAIRE's theorem shows that F contains an element f having property (i). See HAJEK [1971] for details (where Lemma 17 of HAJEK [1971] should be complemented by CHEN [1975]). □

3.11. REMARKS. We cannot apply Theorem 3.4 in order to be able to assume without restriction of generalty that X is compact: we have no information about the invariant points in possible compactifications of X. Even the passage from X to its one point compactification in the case that X is locally compact (i.e. adding one invariant point) may raise the dimension of S.

The same proof as above works if S is homeomorphic with a closed subset of $\mathbb{R}^N$ for some integer N; then $\langle G,X,\pi\rangle$ can be equivariantly embedded in $\langle G,C_c(G,\mathbb{R}^N),\rho\rangle$; see HAJEK [1971]. For the case G = $\mathbb{R}$ and X compact metric, it was shown in TOADER [1974] that $\mathbb{R}^N$ can be replaced by an arbitrary Banach space; the proof is, up to some obvious modifications (e.g. using DUGUNDJI's extension theorem instead of TIETZE's lemma) an almost literal translation of KAKUTANI [1968]. A similar modification could probably be made for theorem 3.10 in its general form, but it must be admitted that the result obtained so (equivariant embedding in $\langle G,C_c(G,V),\rho\rangle$ if the set S of invariant points can be embedded in V) is hardly preferable over the much simpler construction indicated at the beginning of this Section in 3.1.

For a version of the BEBUTOV-KAKUTANI-HAJEK theorem for local dynamical systems, see CARLSON [1972a].

The next result which we present is due to CARLSON [1972b]. It employs the linear $\mathbb{R}$-space $\langle\mathbb{R},C_\nu^\infty,\tau\rangle$ which is defined by a partial differential equation (cf. Section 1, Example 3). The result states, roughly, that the action of $\mathbb{R}$ on a separable metric space can be obtained as solutions of a partial differential equation.

3.12. THEOREM. *Every $\mathbb{R}$-space $\langle\mathbb{R},X,\pi\rangle$ with X a separable metric space can equivariantly be embedded in the linear $\mathbb{R}$-space $\langle\mathbb{R},C_\nu^\infty,\tau\rangle$.*

PROOF. By Theorem 3.4 above, X may supposed to be a compact metric space, so we need only to construct an equivariant continuous injection of X into C_ν^∞. First, apply 3.3 so as to obtain an equivariant embedding

$$x \mapsto (f_n \circ \pi_x)_{n\in\mathbb{N}} : \langle \mathbb{R}, X, \pi\rangle \to \langle \mathbb{R}, C_c(\mathbb{R}, \mathbb{R}^{\mathbb{N}}), \rho\rangle,$$

where the mappings f_n define an embedding of X into $[0,1]^{\mathbb{N}}$. Now the functions $f_n \circ \pi_x$, $n \in \mathbb{N}$, are glued together to a mapping $x \mapsto g_x : X \to \mathbb{R}^{\mathbb{R}\times\mathbb{R}}$, as follows:

$$g_x(s,t) := \begin{cases} f_n \circ \pi_x(2t-s) & \text{if } 3n-3 \le s-t \le 3n-2 \\ 0 & \text{otherwise.} \end{cases}$$

For an averaging procedure, transforming the functions g_x into elements $\bar{g}_x \in C_\nu^\infty$, we refer to CARLSON [1972b]. The resulting mapping $x \mapsto \bar{g}_x : X \to C_\nu^\infty$ turns out to be an equivariant continuous injection. As X was supposed to be compact, it is actually an embedding. □

3.13. <u>REMARK</u>. In CARLSON [1972b] our Theorem 3.4 was not yet available, and in that paper an additional condition on the action π of $\mathbb{R}$ on X was required in order to guarantee that $x \mapsto \bar{g}_x$ is an embedding.

We shall give now a sort of generalization of the method, used in 3.12 for the case of a group G other than $\mathbb{R}$; see also DE VRIES [1975a].

3.14. <u>THEOREM</u>. *Let* $\langle G,X,\pi\rangle$ *be a ttg with* G *an infinite locally compact topological group and* X *a Tychonov space of weight* $w(X) \le L(G)$, *the Lindelöf degree of* G. *Then* $\langle G,X,\pi\rangle$ *can equivariantly be embedded in the linear* G*-space* $\langle G, C_c(G\times G, \mathbb{R}), r\rangle$ *of Example* 5 *in Section* 1.

<u>PROOF</u>. Although the result is also valid if G is compact, we shall assume for convenience that G is not compact. Then there exists a family of continuous functions $\{\psi_i \mid i \in I\}$ from G into the interval [0,1] such that I has cardinality $L(G)$, and

(i) For every $i \in I$ there is $t_i \in G$ such that $\psi_i(t_i) = 1$;

(ii) The supports of the functions ψ_i for $i \in I$ form a disjoint locally finite family of closed subsets of G.

Cf. DE VRIES [1975a], p.115. Now define $\Gamma: \mathbb{R}^I \to \mathbb{R}^G$ by

$$\Gamma(\xi)(t) := \begin{cases} \psi_i(t)\xi_i & \text{if } t \in \text{support } \psi_i \\ 0 & \text{if } t \notin \bigcup_{j\in I} \text{support } \psi_j \end{cases}$$

for $\xi = (\xi_i)_{i\in I} \in \mathbb{R}^I$ and $t \in G$. Using the properties (i) and (ii) of the family $\{\psi_i \mid i \in I\}$ one shows readily that $\Gamma(\xi) \in C(G,\mathbb{R})$ for every $\xi \in \mathbb{R}^I$, and that Γ is, in fact, a topological embedding of $\mathbb{R}^I$ into $C_c(G,\mathbb{R})$ (the rather easy proof appears, somewhat obscured by other arguments, in DE VRIES [1975c], Section 7.2). Now Γ induces an equivariant embedding of $\langle G,C_c(G,\mathbb{R}^I),\rho\rangle$ into $\langle G,C_c(G,C_c(G,\mathbb{R})),\rho\rangle$. Since the latter G-space is isomorphic to $\langle G,C_c(G\times G,\mathbb{R}),r\rangle$ (use DUGUNDJI [1966], XII.5.3 and the fact that G is locally compact), our theorem follows immediately from 3.3. □

3.15. The proof above has been motivated by the following well-known embedding of $[0,1]^{\mathbb{Z}}$ into $C_u([0,1],\mathbb{R})$. Let $n \mapsto a_n\colon \mathbb{Z} \to [0,1]$ be an order preserving injection such that $a_n \to 1$ for $n \to \infty$ and $a_n \to 0$ for $n \to -\infty$. For $\xi = (\xi_n)_{n\in\mathbb{Z}} \in [0,1]^{\mathbb{Z}}$, let $\bar{\xi} \in C([0,1],\mathbb{R})$ be the function obtained by linear interpolation from the values $\bar{\xi}(a_n) := |n|^{-1}\xi_n$ at the points a_n and $\bar{\xi}(0) = \bar{\xi}(1) = 0$. Then $\xi \mapsto \bar{\xi}\colon [0,1]^{\mathbb{Z}} \to C_u([0,1],\mathbb{R})$ is a topological embedding, inducing an equivariant embedding of $\langle G,C_c(G,[0,1]^{\mathbb{Z}}),\rho\rangle$ into the linear G-space $\langle G,C_c(G,C_u([0,1],\mathbb{R})),\rho\rangle$. The latter is isomorphic with $\langle G,C_c(G\times[0,1],\mathbb{R}),r\rangle$, where $r^t f(s,a) := f(st,a)$ for $t \in G$, $f \in C_c(G\times[0,1],\mathbb{R})$ and $(s,a) \in G \times [0,1]$. Together with 3.3 this result implies that *every separable metric G-space* $\langle G,X,\pi\rangle$ *can equivariantly be embedded in the linear G-space* $\langle G,C_c(G,C_u([0,1],\mathbb{R})),\rho\rangle \approx \langle G,C_c(G\times[0,1],\mathbb{R}),r\rangle$. Originally, this result is due to ROŽKO & ŠČERBAKOV [1968].

If G is sigma-compact, then $C_c(G,\ell^2)$ is a complete metrizable locally convex topological vector space (i.e. a *Fréchêt space*). Thus, by 3.2, every separable metrizable G-space $\langle G,X,\pi\rangle$ can equivariantly be embedded in a linear Fréchêt G-space if G is sigma-compact. However, in that case we can even obtain an equivariant embedding into a linear *Hilbert* G-space. To this end, we shall need the linear Hilbert G-spaces $\langle G,L^2(G,H;\nu),\rho\rangle$, defined in Example 7 of Section 1.

3.16. THEOREM. *Let* G *be a sigma-compact group, and let* H *be a Hilbert space of weight* κ. *Then every* G*-space* $\langle G,X,\pi\rangle$ *with* X *a metrizable space of weight* $w(X) \leq \kappa$ *can equivariantly be embedded in the linear Hilbert* G*-space* $\langle G,L^2(G,H;\nu),\rho\rangle$.

PROOF. The space X can be regarded as a subset of the unit ball $\{\xi \mid \xi \in H \ \& \ \|\xi\| \leq 1\}$ of H. This induces an equivariant embedding $\underline{\pi}$: $x \mapsto \pi_x: \langle G,X,\pi\rangle \to \langle G,C_c(G,H),\rho\rangle$ (cf.3.1). Since for every $x \in X$, π_x maps G into the unit ball of H, it follows that $\pi_x \in L^2(G,H;\nu)$, because ν is a bounded measure and π_x is continuous. Since the support of ν is all of G, it is obvious that $\underline{\pi}: X \to L^2(G,H;\nu)$ is injective. Continuity is shown as follows. Let $x \in X$ and $\varepsilon > 0$. By regularity of the measure ν there exists a compact set K in G such that $\nu(G\sim K) < \varepsilon^2/8$. A standard compactness argument shows that there exists a neighbourhood U of x such that $\|\pi(t,x)-\pi(t,y)\| < \varepsilon(2\nu(K))^{-1/2}$ for all $t \in K$ and $y \in U$. Now

$$\|\pi_x-\pi_y\|^2 \leq \int_{G\sim K} \|\pi_x(t)-\pi_y(t)\|^2 d\nu(t) + \int_K \|\pi_x(t)-\pi_y(t)\|^2 d\nu(t)$$

$$\leq 4\nu(G\sim K) + \frac{\varepsilon^2}{2\nu(K)}\nu(K) < \varepsilon^2$$

for all $y \in U$. So π is a continuous equivariant injection of $\langle G,X,\pi\rangle$ into $\langle G,L^2(G,H;\nu),\rho\rangle$. In order to prove that $\underline{\pi}$ is an embedding, consider $x \in X$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\pi_{x_n} \to \pi_x$ in $L^2(G,H;\nu)$, i.e., $\lim_{n\to\infty} \int_G \|\pi_{x_n}(t) - \pi_x(t)\|^2 d\nu(t) = 0$. Then every subsequence of $(x_n)_{n\in\mathbb{N}}$ has a sub-subsequence $(x_{n_i})_{i\in\mathbb{N}}$ with $\lim_{i\to\infty} \|\pi(t,x_{n_i}) - \pi(t,x)\|^2 = 0$ for ν-almost every $t \in G$. Since every π^t is a homeomorphism, it follows that $x_{n_i} \to x$. Consequently, $x_n \to x$ for $n \to \infty$. □

3.17. COROLLARY. *If* G *is compact, then every metrizable* G*-space* $\langle G,X,\pi\rangle$ *can equivariantly be embedded in a unitary Hilbert* G*-space*.

PROOF. In this case, we can take $\nu = \mu$ = right Haar measure. The unitary G-space in which $\langle G,X,\pi\rangle$ can be equivariantly embedded is $\langle G,L^2(G,H),\rho\rangle$; cf. Section 1, Example 6. □

3.18 REMARKS. Constructions of the type, exhibited in the proof of 3.16 can be found at several places in the literature. The trick is, of course, the existence of the measure ν on G, i.e., the existence of the weight function w on G, as shown in Example 7 of Section 1. The proof given above is a modification of DE VRIES [1972] which, in turn, was based on results of BAAYEN & DE GROOT [1968]. For G = $\mathbb{Z}$, a similar method was used in WILLIAMS [1976]. Corollary 3.17 is a modification of the well-known result that every representation of a compact group on a Hilbert space is equivalent with a unitary representation; it appears also in DE GROOT [1962] and in PALAIS [1961]. In the latter paper it was proved by a rather complicated method, using so-called *slices*; see also 3.19 below.

If G is not compact, one may ask whether a metrizable G-space $\langle G,X,\pi\rangle$ can be equivariantly embedded in a unitary Hilbert G-space. Some additional conditions seem necessary because an affirmative answer would imply the existence of an invariant metric on X. Our final result is a slight generalization of PALAIS [1961], where the result has been proved for the case that G is a Lie group. First, we need some definitions.

A set A in a ttg $\langle G,X,\pi\rangle$ is called a *small* set if every point $x \in X$ has a neighbourhood U such that

$$(U,A) := \{t \in G \mid \pi^t U \cap A \neq \emptyset\}$$

has a compact closure in G. The ttg $\langle G,X,\pi\rangle$ is said to be *proper* whenever every point $x \in X$ has a small neighbourhood (this definition differs from the one in BOURBAKI [1971]; for a discussion of the difference, cf. ABELS [1974],1.6). If H is a subgroup of G and $\langle H,X,\pi\rangle$ is a ttg, then an action τ of G on $G \times X$ can be defined by $\tau^t(s,x) := (st^{-1},\pi^t x)$ for $t \in H$, and $(s,x) \in G \times X$. The orbit space of $\langle H,G\times X,\tau\rangle$ (i.e. the quotient space of $G \times X$, obtained by identifying points which lie in the same orbit) will be denoted by $G\times_H X$ and the canonical quotient mapping of $G \times X$ onto $G\times_H X$ by $(s,x) \mapsto [s,x]$. Then by $\tilde{\pi}^t[s,x] := [ts,x]$, $t \in G$ and $[s,x] \in G\times_H X$, an action $\tilde{\pi}$ of G on $G\times_H X$ is defined such that the mapping $x \mapsto [e,x]: X \to G\times_H X$ is H-equivariant. For categorical properties of this construction, see DE VRIES [1979b].

3.19. <u>THEOREM</u>. *Let $\langle G,X,\pi\rangle$ be a proper ttg with X a separable metrizable space. Then there exists an equivariant embedding of $\langle G,X,\pi\rangle$ in a unitary Hilbert G-space $\langle G,\mathcal{H},\sigma\rangle$.*

PROOF. (outline). The proof is a modification of Section 4.3 of PALAIS [1961]. It consists of the following steps.

1. If H is a compact subgroup of G then every unitary Hilbert H-space $\langle H, \mathcal{H}, \sigma \rangle$ admits an H-equivariant embedding in a unitary Hilbert G-space $\langle G, \mathcal{H}', \sigma' \rangle$. This is exactly Lemma 1 of PALAIS [1961], Section 4.3. The proof uses general representation theory of compact groups and the Stone-Weierstrass theorem.

2. If H is a compact subgroup of G and a G_δ-subset of G, and if the H-space $\langle H, S, \pi \rangle$ can H-equivariantly be embedded in the unitary G-space $\langle G, \mathcal{H}', \sigma' \rangle$, then the twisted product $\langle G, G \times_H S, \tilde{\pi} \rangle$ can G-equivariantly be embedded in the unitary Hilbert G-space $L^2(G) \oplus \mathcal{H}'$. A proof can be given, based on ideas from PALAIS [1961], 2.1.4 and Section 4.3.

3. If H is a compact subgroup of G and a G_δ-subset of G, then for every metrizable H-space $\langle H, S, \pi \rangle$ there exists an equivariant embedding of $\langle G, G \times_H S, \tilde{\pi} \rangle$ in a unitary Hilbert G-space. This is an immediate consequence of 1, 2 and 3.17.

4. In order to prove the theorem, we may assume without restriction of generality that G acts *effectively* on X. Since X is separable and metrizable, this implies that G is metrizable, and every compact subgroup of G is G_δ-set in G. By a result of ABELS [1978], X can be covered by invariant open subsets, each of the form $\langle G, G \times_H S, \tilde{\pi} \rangle$ for some compact subgroup H of G. By 3, each of them can equivariantly be embedded in a unitary Hilbert G-space. Using the technique, used in the proof of Lemma 3 of PALAIS [1961], Section 4.3, the proof can be completed. □

3.20. Also in ergodic theory there are results about embeddings of certain systems in "shift systems" which are actually subsystems of $\langle \mathbb{R}, C_c(\mathbb{R}), \rho \rangle$. See e.g. EBERLEIN [1973] and the references given there.

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